

Generalised root identities for zeta functions of curves over finite fields

Richard Stone

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Abstract

We consider generalised root identities for zeta functions of curves over finite fields, ζ_k , and compare with the corresponding analysis for the Riemann zeta function. We verify numerically that, as for ζ , the ζ_k do satisfy the generalised root identities and we investigate these in detail for the special cases of $\mu = 0, -1$ & -2 . Unlike for ζ , however, we show that in the setting of zeta functions of curves over finite fields the $\mu = -2$ root identity is consistent with the Riemann hypothesis (RH) proved by Weil. Comparison of this analysis with the corresponding calculations for ζ illuminates the fact that, even though both ζ and ζ_k have both Euler and Hadamard product representations, it is the detailed structure of the counting function, $N(T)$, which drives the Cesaro computations on the root side of these identities and thereby determines the implications of the root identities for RH in each setting.

1 Introduction

In [1] and [2] it was shown that the Riemann zeta function satisfies the generalised root identities, namely:

$$\frac{-1}{\Gamma(\mu)} \left(\frac{d}{ds} \right)^\mu (\ln(\zeta(s)))|_{s=s_0} = e^{i\pi\mu} \sum_{\{s_0 - \text{roots } r_i \text{ of } \zeta\}} \frac{M_i}{(s_0 - r_i)^\mu} \quad (1)$$

for all $\Re(s_0) > 1$ and arbitrary $\mu \in \mathbb{R}$ (and hence arbitrary $\mu \in \mathbb{C}$ by analytic continuation). In this setting the derivative side is defined by the Euler product formula, leading to an explicit expression already convergent for arbitrary $\mu \in \mathbb{C}$; and in the case of $\mu = 1$ the root identity (1) is equivalent to the Hadamard product formula for the closely related function ξ (at least after “renormalisation” and removal of an obstruction - see [1] and [3, page 35]).

Explicit calculation of the values on the root side at $\mu = 0, -1$ & -2 using generalised Cesaro methods was then performed in [1] and led, in the case of $\mu = -2$, to a claimed contradiction of the Riemann hypothesis (RH).

Given the nature of this claim, it is interesting to consider related settings in which the same techniques may be applied; one such setting is the class of

zeta functions of curves over finite fields, in which both Euler and Hadamard product formulas hold, but the corresponding RH is famously known to be true.

The purpose of this paper is to carry out the analogous root identity computations for this class of zeta functions. By doing so we are able to confirm that the root identities approach is both applicable (these zeta functions satisfy the generalised root identities (1), just like ζ) and consistent with the truth of the RH for such zeta functions.

In turn these computations allow us to isolate partially why the RH is not contradicted by the $\mu = -2$ root identity in the setting of curves over finite fields, unlike for ζ . Roughly, we find that it is not merely the existence of both Hadamard and Euler product formulas that is relevant, but critically also the detailed breakdown of the counting function $N(T)$ (counting non-trivial roots with imaginary part in $[0, T)$) into divergent, oscillatory and decaying asymptotic pieces ($\tilde{N}(T)$, $S(T)$ and $\delta(T)$ respectively).

1.1 Overview

In section 2 we recall the general form of a zeta function, ζ_k , over a finite field k and use it to deduce both (a) the general form of the derivative side $d_{\zeta_k}(s_0, \mu)$ of the root identities for ζ_k and (b) the location of the roots of ζ_k and hence the form of the sum expressing $r_{\zeta_k}(s_0, \mu)$.

Based on this we then show that confirming the root identities for general ζ_k can be reduced to verifying a particular sub-identity in which all the generalised roots (i.e. roots or poles here) lie in an equally spaced fashion on a vertical line $\Re(s) = \sigma_0$ with $0 \leq \sigma \leq 1$ in the complex plane.

Restricting attention to $\Re(s_0) > 1$ as usual, for $\mu > 1$ we then directly verify this sub-identity numerically using the fact that the resulting sum on the root side is classically convergent. And since the roots are equally spaced, we then use the Euler-McLaurin sum formula to extend this readily also to arbitrary $\mu \in \mathbb{R}_{<1}$ by “throwing away” the divergences which then arise, without any need for additional Cesaro averaging.

It is thus verified that, for arbitrary zeta functions over finite fields, the generalised root identities

$$d_{\zeta_k}(s_0, \mu) = r_{\zeta_k}(s_0, \mu) \quad \forall \Re(s_0) > 1$$

are indeed satisfied for arbitrary $\mu \in \mathbb{R}$ (and hence arbitrary $\mu \in \mathbb{C}$ by analytic continuation).

In section 3 we then turn to considering the cases of the $\mu = 0, -1$ & -2 root identities directly, as per [1] for ζ . It is readily shown that $d_{\zeta_k}(s_0, \mu) = 0$ as a function of s_0 whenever $\mu \in \mathbb{Z}_{\leq 0}$, just as for ζ . And by again reducing on the root side to the simpler case of a single vertical line, $\Re(s) = \sigma_0$, with equally spaced roots, we are able to deduce by generalised Cesaro means that we likewise have

$$r_{\zeta_k}(s_0, 0) = r_{\zeta_k}(s_0, -1) = r_{\zeta_k}(s_0, -2) = 0$$

as functions of s_0 . Hence we validate explicitly the $\mu = 0, -1$ & -2 root identities and we see, moreover, that unlike for ζ , the $\mu = -2$ root identity for ζ_k could not lead to a contradiction of RH in this setting since the Cesaro results for equi-spaced roots on a vertical line hold equally whether $\sigma_0 = \frac{1}{2}$ or $\sigma_0 \neq \frac{1}{2}$.

To confirm this more directly, however, in section 4 we re-perform the calculation of the root sides $r_{\zeta_k}(s_0, \mu)$ for $\mu = 0, -1$ & -2 in a way which directly mimics the calculations in [1, section 4.2]. We again find $r_{\zeta_k}(s_0, 0) = r_{\zeta_k}(s_0, -1) = 0$ and $r_{\zeta_k}(s_0, -2) = X_\epsilon$, using the notation of [1] where the corresponding formula for $\mu = -2$ was $r_\zeta(s_0, -2) = -\frac{1}{2} + X_\epsilon$. The absence of the $-\frac{1}{2}$ term for the ζ_k case confirms again that, for the setting of zeta functions of curves over finite fields, the $\mu = -2$ root identity thus does not contradict the RH for such zeta functions.

We end the paper with some final observations. Firstly we conclude section 4 by noting why the $\mu = -2$ root identity, which thus requires $X_\epsilon = 0$, cannot in fact be used to actually *deduce* the RH for such ζ_k . Then, after summarising in section 5, we consider fundamental differences in the nature of the counting functions, $N(T)$, between the cases of ζ_k and ζ , and discuss how these differences are driving the divergences in the calculations for $\mu = 0, -1$ & -2 in the two settings (and thereby the consequences for RH) despite having both Euler and Hadamard product formulas in both cases.

2 The Root Identities for Zeta Functions of Curves over Finite Fields

We adopt the notation of the exposition in [3, Chapter 5]; F_q is the finite field with q elements, where $q = p^n$, p prime; and k is a finitely-generated extension of F_q of transcendence degree 1, so that k is the algebraic extension of $F_q(x)$ generated by y satisfying $G(x, y) = 0$ for some irreducible polynomial $G \in F_q[x, y]$. Thus k may be viewed as the field of meromorphic functions on the curve C defined by $G(x, y) = 0$.

The zeta function associated to k and C is then defined by the Euler product formula

$$\zeta_k(s) = \prod_{w \in \Sigma(k)} (1 - q^{-f(w)s})^{-1} \quad (2)$$

which converges classically in the half-plane $\Re(s) > 1$; here $\Sigma(k)$ is the set of places of k (including ∞) and $f : \Sigma(k) \rightarrow \mathbb{Z}_{>0}$ defines the degree of each $w \in \Sigma(k)$. As for ζ , (2) implies that ζ_k has no poles or roots in $\{s : \Re(s) > 1\}$.

For purposes of investigating the generalised root identities for ζ_k , however, the starting point is the fact that ζ_k , analytically continued to all of \mathbb{C} , has the form

$$\zeta_k(s) = \frac{P_{2g}(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} \quad (3)$$

where P_{2g} is a polynomial with real coefficients of degree $2g$, g being the genus of the curve C . P_{2g} has the property that it may be factorised as

$$P_{2g}(u) = \prod_{\lambda \in A} (1 - \lambda u) \quad (4)$$

where the set A is closed under the mapping $\lambda \rightarrow \frac{q}{\lambda}$.¹ It follows from this and the functional equation for ζ_k

$$\zeta_k(1-s) = q^{(1-g)(1-2s)} \zeta_k(s) \quad (5)$$

that the generalised roots (roots and poles) of ζ_k all lie in the critical strip $0 \leq \Re(s) \leq 1$ and, like for ζ , are mirror-symmetric in both the real axis and the critical line $\Re(s) = \frac{1}{2}$.

The restriction of the generalised roots to the critical strip reflects that $1 \leq |\lambda| \leq q$ for all the factors in $P_{2g}(u)$ in (4) and if we add to A the two terms $\lambda = 1$ and $\lambda = q$ corresponding to the two factors in the denominator in (3), and thus define

$$\tilde{A} = A \cup \{1, q\} \quad (6)$$

then we see that ζ_k may be written in (3) as

$$\zeta_k(s) = \prod_{\lambda \in \tilde{A}} (1 - \lambda q^{-s})^{\nu_\lambda} \quad (7)$$

where $\nu_\lambda = -1$ for $\lambda \in \{1, q\}$ and $\nu_\lambda = 1$ otherwise.

The RH for zeta functions of curves over finite fields is the claim that, for any such ζ_k ,

$$|\lambda| = q^{\frac{1}{2}} \quad \forall \lambda \in A \quad (8)$$

or equivalently that all the actual roots (not poles) of ζ_k lie strictly on the critical line $\Re(s) = \frac{1}{2}$.

The expression (7) is immediately well-adapted for analysing the generalised root identities (1) for ζ_k . In order to do this, from this point on we restrict consideration to $\Re(s_0) > 1$.

On the derivative side we immediately have

¹Here is the only place where we amend the notation of [3] to use λ rather than α since we wish to reserve α for its usual role in later Cesaro calculations.

$$\begin{aligned}
d_{\zeta_k}(s_0, \mu) &= \frac{-1}{\Gamma(\mu)} \sum_{\lambda \in \tilde{A}} \nu_\lambda \left(\frac{d}{ds} \right)^\mu (\ln(1 - \lambda q^{-s}))|_{s=s_0} \\
&= \frac{1}{\Gamma(\mu)} \sum_{\lambda \in \tilde{A}} \nu_\lambda \left(\frac{d}{ds} \right)^\mu \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n q^{-ns} \right\} |_{s=s_0} \\
&= \frac{1}{\Gamma(\mu)} \sum_{\lambda \in \tilde{A}} \nu_\lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n} e^{i\pi\mu n^\mu (\ln q)^\mu q^{-ns_0}} \\
&= \frac{e^{i\pi\mu}}{\Gamma(\mu)} (\ln q)^\mu \sum_{\lambda \in \tilde{A}} \nu_\lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{1-\mu}} q^{-ns_0} \tag{9}
\end{aligned}$$

which is convergent for arbitrary μ since $|\lambda q^{-s_0}| < 1$.

On the root side, for any given $\lambda \in \tilde{A}$, in (7) we clearly have infinitely many generalised roots of ζ_k associated to the factor $(1 - \lambda q^{-s})^{\nu_\lambda}$; if we write

$$\lambda = q^{\sigma_0} e^{i\theta_0} \quad \text{where} \quad 0 \leq \sigma_0 \leq 1, \quad 0 \leq \theta_0 < 2\pi \tag{10}$$

then these are all on the vertical line $\Re(s_0) = \sigma_0$. There is a unique such root having imaginary part in $[0, \frac{2\pi}{\ln q})$, which we shall call the “base root” of ζ_k associated to this λ -factor and which we shall denote $r_0^{(\lambda)}$ so that

$$r_0^{(\lambda)} = \sigma_0 + i\tau_0 \quad \text{where} \quad \tau_0 = \frac{\theta_0}{\ln q} \tag{11}$$

The other roots associated to this λ -factor are equally spaced up and down this vertical line in the complex plane at intervals of

$$C := \frac{2\pi}{\ln q} \tag{12}$$

i.e the roots of ζ_k arising from the factor $(1 - \lambda q^{-s})^{\nu_\lambda}$ in (7) are the set

$$R_\lambda := \{r_j\}_{j \in \mathbb{Z}} \quad \text{where} \quad r_j = r_0^{(\lambda)} + i \cdot Cj = \sigma_0 + i(\tau_0 + Cj) \tag{13}$$

If $\nu_\lambda = -1$ these are the poles (all with multiplicity $M_i = -1$) associated to the two factors on the denominator in (3) and lying on the boundary-lines of the critical strip $\Re(s_0) = 0$ and $\Re(s_0) = 1$; if $\nu_\lambda = 1$ they are the roots (all with multiplicity $M_i = 1$) associated to the factors of P_{2g} .

On the root side of the generalised root identities (1) we then have, on noting $\nu_\lambda = M_i$ in all cases, that

$$\begin{aligned}
r_{\zeta_k}(s_0, \mu) &= e^{i\pi\mu} \sum_{\lambda \in \bar{A}} \nu_\lambda \sum_{r_j \in R_\lambda} \frac{1}{(s_0 - r_j)^\mu} \\
&= e^{i\pi\mu} \sum_{\lambda \in \bar{A}} \nu_\lambda \sum_{j=-\infty}^{\infty} \frac{1}{((s_0 - r_0^{(\lambda)}) - i \cdot Cj)^\mu} \quad (14)
\end{aligned}$$

which is classically convergent for $\Re(\mu) > 1$ and can then be analytically continued to $\Re(\mu) \leq 1$.

Confirming whether ζ_k satisfies the generalised root identities thus consists of testing whether the expressions for $d_{\zeta_k}(s_0, \mu)$ and $r_{\zeta_k}(s_0, \mu)$ in (9) and (14) agree for arbitrary $\Re(s_0) > 1$ and arbitrary $\mu \in \mathbb{R}$ (and hence arbitrary $\mu \in \mathbb{C}$ by analytic continuation). It is clear, in turn, from these expressions that this will be true if the contributions on each side from each λ -factor $(1 - \lambda q^{-s})^{\nu_\lambda}$ agree; i.e. if

$$d_{\zeta_k}^{(\lambda)}(s_0, \mu) = r_{\zeta_k}^{(\lambda)}(s_0, \mu) \quad (15)$$

where

$$d_{\zeta_k}^{(\lambda)}(s_0, \mu) = \frac{e^{i\pi\mu}}{\Gamma(\mu)} (\ln q)^\mu \nu_\lambda \sum_{n=1}^{\infty} \frac{\lambda^n}{n^{1-\mu}} q^{-ns_0} \quad (16)$$

and

$$r_{\zeta_k}^{(\lambda)}(s_0, \mu) = e^{i\pi\mu} \nu_\lambda \sum_{j=-\infty}^{\infty} ((s_0 - r_0^{(\lambda)}) - i \cdot Cj)^{-\mu} \quad (17)$$

and where λ and $r_0^{(\lambda)}$ are related by (10) and (11).

For $\mu \in \mathbb{R}_{>1}$, since both sides are convergent, this may be checked directly numerically. For $\mu \in \mathbb{R}_{\leq 1}$ the Euler-McLaurin sum formula (e.g. as stated in [4]) may be used to calculate the analytic continuation on the root side explicitly, since the roots contributing to $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ are evenly spaced. We have

$$\sum_{j=1}^k ((s_0 - r_0^{(\lambda)}) - i \cdot Cj)^{-\mu} = \left\{ \begin{array}{l} \frac{i}{(1-\mu)C} ((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{1-\mu} \\ + f_+(s_0, \mu) \\ + \frac{1}{2} ((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu} \\ + \frac{i\mu C}{12} ((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu-1} \\ + \frac{i\mu(\mu+1)(\mu+2)C^3}{720} ((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu-3} \\ + \dots \end{array} \right\} \quad (18)$$

and similarly

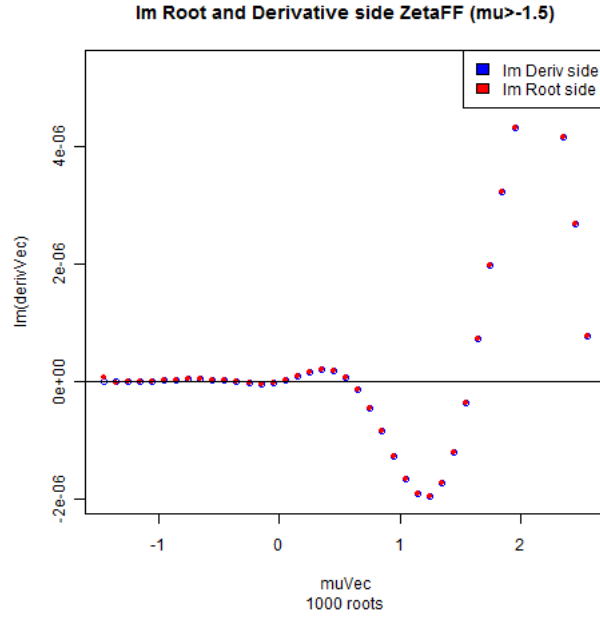
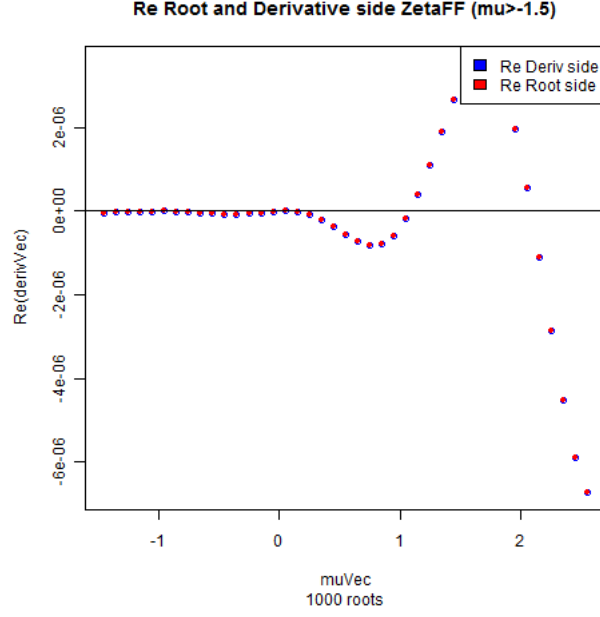
$$\sum_{j=1}^{\tilde{k}} ((s_0 - r_0^{(\lambda)}) + i \cdot Cj)^{-\mu} = \left\{ \begin{array}{l} -\frac{i}{(1-\mu)C} ((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{1-\mu} \\ + f_-(s_0, \mu) \\ + \frac{1}{2} ((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu} \\ - \frac{i\mu C}{12} ((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu-1} \\ - \frac{i\mu(\mu+1)(\mu+2)C^3}{720} ((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu-3} \\ + \dots \end{array} \right\} \quad (19)$$

and since $f_{\pm}(s_0, \mu)$ clearly represent the analytic continuations of these sums from $\mu \in \mathbb{R}_{>1}$ to $\mu \in \mathbb{R}_{\leq 1}$, so in general

$$\begin{aligned}
r_{\zeta_k}^{(\lambda)}(s_0, \mu) &= \lim_{k, \tilde{k} \rightarrow \infty} e^{i\pi\mu} \nu_\lambda \\
&\times \left[\begin{aligned} &\sum_{j=-\tilde{k}}^k ((s_0 - r_0^{(\lambda)}) - i \cdot Cj)^{-\mu} - \\ &(s_0 - r_0^{(\lambda)})^{-\mu} - \\ &\frac{i}{(1-\mu)C} \left\{ \begin{aligned} &((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{1-\mu} - \\ &((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{1-\mu} \end{aligned} \right\} - \\ &\frac{1}{2} \left\{ ((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu} + ((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu} \right\} - \\ &\frac{i\mu C}{12} \left\{ \begin{aligned} &((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu-1} - \\ &((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu-1} \end{aligned} \right\} - \\ &\frac{i\mu(\mu+1)(\mu+2)C^3}{720} \left\{ \begin{aligned} &((s_0 - r_0^{(\lambda)}) - i \cdot Ck)^{-\mu-3} - \\ &((s_0 - r_0^{(\lambda)}) + i \cdot C\tilde{k})^{-\mu-3} \end{aligned} \right\} \\ &- \dots \end{aligned} \right] \quad (20)
\end{aligned}$$

where the number of divergent terms to be subtracted is dictated by the value of μ and we may as well take $\tilde{k} = k$.

The formulae for $d_{\zeta_k}^{(\lambda)}(s_0, \mu)$ and $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ in (16) and (20) have been implemented in R-code and their equality checked for a variety of values of λ , s_0 and $\mu \in \mathbb{R}$. For example the graphs below show the equality of $d_{\zeta_k}^{(\lambda)}(s_0, \mu)$ and $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ for $s_0 = 5.1238$ and a variety of μ -values ranging from $\mu = -1.5$ to $\mu = 2.6$ in the case when $q = 5^2$ and λ is given by $\sigma_0 = 0.6$, $\tau_0 = \frac{3\pi}{4}$; here we have used $k = \tilde{k} = 1,000$ roots on the root side in (20) in calculating the approximation to $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$, while we have approximated $d_{\zeta_k}^{(\lambda)}(s_0, \mu)$ in (16) using 20 terms in the sum for each μ . The R-code generating these graphs is given in Appendix 1.



Based on these sort of numerical verifications we are able to conclude that:

Result 1: *We have*

$$d_{\zeta_k}^{(\lambda)}(s_0, \mu) = r_{\zeta_k}^{(\lambda)}(s_0, \mu) \quad (21)$$

in general for arbitrary s_0 and $\mu \in \mathbb{R}$ (and hence arbitrary $\mu \in \mathbb{C}$ by analytic continuation) for any allowed λ -factor. Hence for any zeta function, ζ_k , of a curve over a finite field k the generalised root identities are also satisfied, i.e.

$$d_{\zeta_k}(s_0, \mu) = r_{\zeta_k}(s_0, \mu) \quad (22)$$

for arbitrary s_0 and $\mu \in \mathbb{R}$ (and hence arbitrary $\mu \in \mathbb{C}$ by analytic continuation).

3 The root identities for ζ_k at $\mu = 0, -1$ and -2

We now turn, as for ζ in [1], to considering the particular cases of these generalised root identities for ζ_k at $\mu = 0, -1$ and -2 .

On the derivative side in (9), since the sum is convergent for arbitrary μ and the factor $\Gamma(\mu)$ on the denominator diverges for $\mu \in \mathbb{Z}_{\leq 0}$, it follows at once that, as for ζ , we have

Result 2: As a function of s_0 on $\Re(s_0) > 1$

$$d_{\zeta_k}(s_0, \mu) = 0 \quad (23)$$

whenever $\mu \in \mathbb{Z}_{\leq 0}$.

On the root side, as in section 2, we use the fact that

$$r_{\zeta_k}(s_0, \mu) = \sum_{\lambda \in \tilde{A}} r_{\zeta_k}^{(\lambda)}(s_0, \mu) \quad (24)$$

to reduce the computations to the case of $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ for any single given λ -factor $(1 - \lambda q^{-s})^{\nu_\lambda}$.

In this case the roots are all equi-spaced vertically at intervals of $C = \frac{2\pi}{\ln q}$ either side of the base root $r_0^{(\lambda)} = \sigma_0 + i\tau_0$ on the line $\Re(s) = \sigma_0$.

Adopting our standard notation for Cesaro calculations we thus have

$$T = Ck + \tau_0 + \alpha \quad \text{and} \quad \tilde{T} = C\tilde{k} - \tau_0 + \tilde{\alpha} \quad (25)$$

where now $\alpha, \tilde{\alpha} \in [0, C)$; and the variables z and \tilde{z} arising from consideration of $\{s_0 - \text{roots } r_i \in R_\lambda\}$ are given by

$$z = (s_0 - \sigma_0) - iT \quad \text{so} \quad T = i(z - (s_0 - \sigma_0)) \quad (26)$$

and

$$\tilde{z} = (s_0 - \sigma_0) + i\tilde{T} \quad \text{so} \quad \tilde{T} = -i(\tilde{z} - (s_0 - \sigma_0)) \quad (27)$$

Recalling our definition of Cesaro convergence along a contour in [1, section 2] (namely that we (a) first remove generalised Cesaro eigenfunctions in the *geometric* variables (z or \tilde{z}) and then (b) apply a suitable power of the Cesaro averaging operator, P , to the residual function to reduce to one with a classical limit) we are then able to deduce the following calculational results which arise in the computation of $r_{\zeta_k}^{(\lambda)}(s_0, 0)$, $r_{\zeta_k}^{(\lambda)}(s_0, -1)$ and $r_{\zeta_k}^{(\lambda)}(s_0, -2)$:

Lemma: (1) *On the contour traversed by $z = (s_0 - \sigma_0) - iT$ we have*
(a)

$$Clim_{z \rightarrow \infty} \alpha^n = \frac{1}{n+1} C^n$$

(b)

$$Clim_{z \rightarrow \infty} k\alpha = -\frac{i}{2}(s_0 - r_0^{(\lambda)}) - \frac{1}{3}C \quad \text{and}$$

$$Clim_{z \rightarrow \infty} k\alpha^2 = -\frac{i}{3}C(s_0 - r_0^{(\lambda)}) - \frac{1}{4}C^2 \quad \text{and}$$

$$Clim_{z \rightarrow \infty} k^2\alpha = -\frac{1}{2C}(s_0 - r_0^{(\lambda)})^2 + \frac{7}{12}i(s_0 - r_0^{(\lambda)}) + \frac{1}{4}C + \frac{1}{12}\tau_0$$

(c)

$$Clim_{z \rightarrow \infty} z\alpha = Clim_{z \rightarrow \infty} z\alpha^2 = 0 \quad \text{and}$$

$$Clim_{z \rightarrow \infty} z^2\alpha = \frac{i}{12}C^2(s_0 - \sigma_0)$$

(d)

$$Clim_{z \rightarrow \infty} k = \frac{1}{C} \left\{ -i(s_0 - r_0^{(\lambda)}) - \frac{1}{2}C \right\} \quad \text{and}$$

$$Clim_{z \rightarrow \infty} k^2 = \frac{1}{C^2} \left\{ -(s_0 - r_0^{(\lambda)})^2 + iC(s_0 - r_0^{(\lambda)}) + \frac{1}{3}C^2 \right\} \quad \text{and}$$

$$Clim_{z \rightarrow \infty} k^3 = \frac{1}{C^3} \left\{ \begin{aligned} &\frac{i}{4}C^2(s_0 - \sigma_0) + i(s_0 - r_0^{(\lambda)})^3 + \frac{3C}{2}(s_0 - r_0^{(\lambda)})^2 \\ &-iC^2(s_0 - r_0^{(\lambda)}) - \frac{1}{4}C^3 \end{aligned} \right\}$$

(2) *Similarly, on the contour traversed by $\tilde{z} = (s_0 - \sigma_0) + i\tilde{T}$ we have*
(a)

$$Clim_{\tilde{z} \rightarrow \infty} \tilde{\alpha}^n = \frac{1}{n+1} C^n$$

(b)

$$Clim_{\tilde{z} \rightarrow \infty} \tilde{k}\tilde{\alpha} = \frac{i}{2}(s_0 - r_0^{(\lambda)}) - \frac{1}{3}C \quad \text{and}$$

$$Clim_{\tilde{z} \rightarrow \infty} \tilde{k}\tilde{\alpha}^2 = \frac{i}{3}C(s_0 - r_0^{(\lambda)}) - \frac{1}{4}C^2 \quad \text{and}$$

$$Clim_{\tilde{z} \rightarrow \infty} \tilde{k}^2\tilde{\alpha} = -\frac{1}{2C}(s_0 - r_0^{(\lambda)})^2 - \frac{7}{12}i(s_0 - r_0^{(\lambda)}) + \frac{1}{4}C - \frac{1}{12}\tau_0$$

(c)

$$\mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{z} \tilde{\alpha} = \mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{z} \tilde{\alpha}^2 = 0 \quad \text{and}$$

$$\mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{z}^2 \tilde{\alpha} = -\frac{i}{12} C^2 (s_0 - \sigma_0)$$

(d)

$$\mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{k} = \frac{1}{C} \left\{ i(s_0 - r_0^{(\lambda)}) - \frac{1}{2} C \right\} \quad \text{and}$$

$$\mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{k}^2 = \frac{1}{C^2} \left\{ -(s_0 - r_0^{(\lambda)})^2 - iC(s_0 - r_0^{(\lambda)}) + \frac{1}{3} C^2 \right\} \quad \text{and}$$

$$\mathcal{Clim}_{\tilde{z} \rightarrow \infty} \tilde{k}^3 = \frac{1}{C^3} \left\{ \begin{aligned} & -\frac{i}{4} C^2 (s_0 - \sigma_0) - i(s_0 - r_0^{(\lambda)})^3 + \frac{3C}{2} (s_0 - r_0^{(\lambda)})^2 \\ & + iC^2 (s_0 - r_0^{(\lambda)}) - \frac{1}{4} C^3 \end{aligned} \right\}$$

Proof: We shall prove only the results in part (1) for the contour traversed by z ; the adaptation of these arguments for the contour traversed by \tilde{z} in part (2) is straightforward.

The result for $\mathcal{Clim}_{z \rightarrow \infty} \alpha^n$ in (1)(a) is immediate since the function $f(T) = \alpha^n$ is periodic with period C and its average value on each period is $\frac{1}{C} \int_0^C \alpha^n d\alpha = \frac{1}{n+1} C^n$.

For the remaining results in (1)(b)-(1)(d) we then calculate successively in the following order:

$$k \rightarrow k\alpha^n \rightarrow z\alpha^n \rightarrow k^2 \rightarrow k^2\alpha^n \rightarrow z^2\alpha^n \rightarrow k^3$$

with the calculation of the Cesaro limit of each relying on the Cesaro limits of the preceding ones in the chain.

Starting with k we have

$$\begin{aligned} k &= \frac{1}{C} (T - \tau_0 - \alpha) = \frac{1}{C} \{ i(z - (s_0 - \sigma_0)) - \tau_0 - \alpha \} \\ &= \frac{1}{C} \left\{ iz - i(s_0 - r_0^{(\lambda)}) - \alpha \right\} \end{aligned} \quad (28)$$

and thus, since z has generalised Cesaro limit 0 and $\alpha \stackrel{C}{\sim} \frac{1}{2} C$ (by (1)(a)), so

$$k \stackrel{C}{\sim} \frac{1}{C} \left\{ -i(s_0 - r_0^{(\lambda)}) - \frac{1}{2} C \right\}$$

as claimed in (1)(d).

Next, consider $k\alpha^n$. We have, in light of (1)(a), that

$$\begin{aligned}
P[k(\alpha^n - \frac{1}{n+1}C^n)] &= \frac{1}{T} \left\{ \sum_{j=0}^{k-1} j \cdot 0 + k \left(\frac{\alpha^{n+1}}{n+1} - \frac{C^n}{n+1} \alpha \right) \right\} \\
&= \frac{1}{Ck + \tau_0 + \alpha} \cdot k \left(\frac{\alpha^{n+1}}{n+1} - \frac{C^n}{n+1} \alpha \right) \\
&= \frac{1}{C} \left(\frac{\alpha^{n+1}}{n+1} - \frac{C^n}{n+1} \alpha \right) \left(1 - \frac{(\tau_0 + \alpha)}{Ck} + \dots \right) \\
&\stackrel{\mathcal{C}}{\sim} \frac{1}{C} \left(\frac{C^{n+1}}{(n+1)(n+2)} - \frac{1}{2} \frac{C^{n+1}}{n+1} \right) \\
&= -\frac{n}{2(n+1)(n+2)} C^n
\end{aligned}$$

It follows that

$$\begin{aligned}
k\alpha^n &\stackrel{\mathcal{C}}{\sim} \frac{C^n}{(n+1)} k - \frac{n}{2(n+1)(n+2)} C^n \\
&\stackrel{\mathcal{C}}{\sim} \frac{C^{n-1}}{(n+1)} \left\{ -i(s_0 - r_0^{(\lambda)}) - \left[\frac{1}{2}C + \frac{n}{2(n+2)}C \right] \right\} \\
&\stackrel{\mathcal{C}}{\sim} -i \frac{C^{n-1}}{(n+1)} (s_0 - r_0^{(\lambda)}) - \frac{1}{(n+2)} C^n \tag{29}
\end{aligned}$$

using the result just derived for the Cesaro limit of k ; and hence for $n = 1$ and $n = 2$ we get the first two results claimed in (1)(b).

From (29) it then follows immediately that

$$\begin{aligned}
z\alpha^n &= ((s_0 - \sigma_0) - iT)\alpha^n = ((s_0 - \sigma_0) - i(Ck + \tau_0 + \alpha))\alpha^n \\
&= (s_0 - r_0^{(\lambda)})\alpha^n - i\alpha^{n+1} - iCk\alpha^n \\
&\stackrel{\mathcal{C}}{\sim} (s_0 - r_0^{(\lambda)})\frac{C^n}{n+1} - i\frac{C^{n+1}}{n+2} - \frac{C^n}{n+1}(s_0 - r_0^{(\lambda)}) + i\frac{C^{n+1}}{n+2} = 0 \tag{30}
\end{aligned}$$

also verifying the first two results of (1)(c).

And thus, noting (28) and invoking the results already derived in (1)(a) and (1)(c) and the fact that the Cesaro eigenfunctions z^2 and z have generalised Cesaro limit 0, we have

$$\begin{aligned}
k^2 &= \frac{1}{C^2} \left\{ \begin{array}{c} -z^2 + 2(s_0 - r_0^{(\lambda)})z - 2iz\alpha \\ -(s_0 - r_0^{(\lambda)})^2 + 2i(s_0 - r_0^{(\lambda)})\alpha + \alpha^2 \end{array} \right\} \\
&\stackrel{\mathcal{C}}{\sim} \frac{1}{C^2} \left\{ -(s_0 - r_0^{(\lambda)})^2 + iC(s_0 - r_0^{(\lambda)}) + \frac{1}{3}C^2 \right\}
\end{aligned}$$

which verifies the second result claimed in (1)(d).

Next we consider $k^2\alpha^n$. Along similar lines to our earlier calculation for $k\alpha^n$ we have that

$$\begin{aligned}
P[k^2(\alpha^n - \frac{1}{n+1}C^n)] &= \frac{1}{Ck + \tau_0 + \alpha} \left\{ \sum_{j=0}^{k-1} j \cdot 0 + k^2 \left(\frac{\alpha^{n+1}}{n+1} - \frac{C^n}{n+1}\alpha \right) \right\} \\
&= \frac{k}{C} \cdot \left(\frac{\alpha^{n+1}}{n+1} - \frac{C^n}{n+1}\alpha \right) \left(1 - \frac{(\tau_0 + \alpha)}{Ck} + \dots \right) \\
&= \frac{1}{C} \left\{ \begin{array}{c} \frac{k\alpha^{n+1}}{n+1} - \frac{C^n}{n+1}k\alpha - \frac{(\tau_0\alpha^{n+1} + \alpha^{n+2})}{C(n+1)} \\ + \frac{C^{n-1}}{(n+1)}(\tau_0\alpha + \alpha^2) + \dots \end{array} \right\}
\end{aligned}$$

Using (29) and the result from (1)(a) it follows that

$$\begin{aligned}
P[k^2(\alpha^n - \frac{1}{n+1}C^n)] &\stackrel{\mathcal{C}}{\sim} \frac{1}{C} \left\{ \begin{array}{c} \frac{1}{(n+1)} \left[-i\frac{C^n}{(n+2)}(s_0 - r_0^{(\lambda)}) - \frac{C^{n+1}}{(n+3)} \right] \\ -\frac{C^n}{n+1} \left[-\frac{i}{2}(s_0 - r_0^{(\lambda)}) - \frac{C}{3} \right] \\ -\frac{1}{C(n+1)} \left[\tau_0\frac{C^{n+1}}{n+2} + \frac{C^{n+2}}{n+3} \right] \\ +\frac{C^{n-1}}{(n+1)} \left[\frac{\tau_0 C}{2} + \frac{C^2}{3} \right] \end{array} \right\} \\
&= \frac{1}{C} \left\{ \begin{array}{c} iC^n \frac{n}{2(n+1)(n+2)}(s_0 - r_0^{(\lambda)}) \\ + \frac{2n}{3(n+1)(n+3)}C^{n+1} \\ + \frac{n}{2(n+1)(n+2)}\tau_0 C^n \end{array} \right\}
\end{aligned}$$

and thus, by the result just proved for the Cesaro limit of k^2 , we have that

$$\begin{aligned}
k^2 \alpha^n &\stackrel{\mathcal{C}}{\sim} \frac{C^n}{n+1} k^2 + P[k^2(\alpha^n - \frac{1}{n+1} C^n)] \\
&\stackrel{\mathcal{C}}{\sim} \left\{ \begin{aligned} &-\frac{C^{n-2}}{(n+1)} (s_0 - r_0^{(\lambda)})^2 \\ &+ i C^{n-1} \frac{(3n+4)}{2(n+1)(n+2)} (s_0 - r_0^{(\lambda)}) \\ &+ \frac{C^n}{(n+3)} + \frac{n}{2(n+1)(n+2)} \tau_0 C^{n-1} \end{aligned} \right\} \quad (31)
\end{aligned}$$

which for $n = 1$ proves the last of the claimed results in (1)(b).

Hence, along the same lines as our previous calculation for $z\alpha^n$, we have

$$\begin{aligned}
z^2 \alpha^n &= ((s_0 - r_0^{(\lambda)}) - i C k - i \alpha)^2 \alpha^n \\
&= \left\{ \begin{aligned} &(s_0 - r_0^{(\lambda)})^2 \alpha^n - C^2 k^2 \alpha^n - \alpha^{n+2} \\ &- 2i(s_0 - r_0^{(\lambda)}) C k \alpha^n - 2C k \alpha^{n+1} - 2i(s_0 - r_0^{(\lambda)}) \alpha^{n+1} \end{aligned} \right\} \\
&\stackrel{\mathcal{C}}{\sim} \frac{n}{2(n+1)(n+2)} C^{n+1} \left\{ i(s_0 - r_0^{(\lambda)}) - \tau_0 \right\} \\
&= \frac{n}{2(n+1)(n+2)} C^{n+1} i(s_0 - \sigma_0) \quad (32)
\end{aligned}$$

after invoking all the Cesaro limits already derived above and extensive cancellations. When $n = 1$ this verifies the last of the claimed results in (1)(c).

This leaves only the Cesaro limit of k^3 to calculate. Here, mimicking the calculation for k^2 and using the results from (1)(a) and (1)(c) and the fact that the Cesaro eigenfunctions z , z^2 and z^3 all have generalised Cesaro limit 0, we have that

$$\begin{aligned}
k^3 &= \frac{1}{C^3} \left\{ \begin{aligned} &-iz^3 + 3z^2[i(s_0 - r_0^{(\lambda)}) + \alpha] \\ &+ 3iz[i(s_0 - r_0^{(\lambda)}) + \alpha]^2 \\ &-[i(s_0 - r_0^{(\lambda)}) + \alpha]^3 \end{aligned} \right\} \\
&\stackrel{\mathcal{C}}{\sim} \frac{1}{C^3} \left\{ \begin{aligned} &\frac{i}{4} C^2 (s_0 - \sigma_0) + i(s_0 - r_0^{(\lambda)})^3 + \frac{3C}{2} (s_0 - r_0^{(\lambda)})^2 \\ &-iC^2 (s_0 - r_0^{(\lambda)}) - \frac{1}{4} C^3 \end{aligned} \right\}
\end{aligned}$$

which verifies the last of the claimed results in (1)(d) and thus completes the proof.

Using this lemma we now calculate $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ for $\mu = 0, -1$ and -2 by Cesaro means. Let $N_{+, \mu}(T)$ be the partial-sum function arising from summation over roots with imaginary parts in $[0, T)$ on the line $\Re(s) = \sigma_0$ in the definition of $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$; and let $N_{-, \mu}(T)$ be the corresponding partial-sum function for the roots with imaginary parts in $(-T, 0)$. Thus $N_{+, \mu}(T)$ is actually the partial-sum function on the contour traversed by $z = (s_0 - \sigma_0) - iT$, given by

$$N_{+, \mu}(T) = \sum_{j=0}^k ((s_0 - r_0^{(\lambda)}) - iCj)^{-\mu} \quad (33)$$

and $N_{-, \mu}(T)$ is the partial-sum function on the contour traversed by $\tilde{z} = (s_0 - \sigma_0) + i\tilde{T}$, given by

$$N_{-, \mu}(\tilde{T}) = \sum_{j=1}^{\tilde{k}} ((s_0 - r_0^{(\lambda)}) + iCj)^{-\mu} \quad (34)$$

It follows at once that, in general,

$$r_{\zeta_k}^{(\lambda)}(s_0, \mu) = e^{i\pi\mu} \nu_\lambda \cdot \lim_{z, \tilde{z} \rightarrow \infty} \left\{ N_{+, \mu}(T) + N_{-, \mu}(\tilde{T}) \right\} \quad (35)$$

Case 1 - $\mu = 0$: In this case

$$N_{+, 0}(T) = k + 1 \stackrel{\mathcal{C}}{\sim} -\frac{i}{C}(s_0 - r_0^{(\lambda)}) + \frac{1}{2}$$

and

$$N_{-, 0}(\tilde{T}) = \tilde{k} \stackrel{\mathcal{C}}{\sim} \frac{i}{C}(s_0 - r_0^{(\lambda)}) - \frac{1}{2}$$

Thus immediately in (35) we have

$$r_{\zeta_k}^{(\lambda)}(s_0, 0) = 0 \quad (36)$$

as a function of s_0 , and therefore in (24)

$$r_{\zeta_k}(s_0, 0) = 0 \quad (37)$$

also as a function of s_0 , so that $d_{\zeta_k}(s_0, 0) = r_{\zeta_k}(s_0, 0)$ and the generalised root identity for ζ_k is confirmed when $\mu = 0$.

Case 2 - $\mu = -1$: Here, after simplification,

$$\begin{aligned} N_{+, -1}(T) &= (s_0 - r_0^{(\lambda)})(k + 1) - i\frac{C}{2}(k^2 + k) \\ &\stackrel{\mathcal{C}}{\sim} -\frac{i}{2C}(s_0 - r_0^{(\lambda)})^2 + \frac{1}{2}(s_0 - r_0^{(\lambda)}) + \frac{i}{12}C \end{aligned}$$

and

$$\begin{aligned}
N_{-,-1}(\tilde{T}) &= (s_0 - r_0^{(\lambda)})\tilde{k} + i\frac{C}{2}(\tilde{k}^2 + \tilde{k}) \\
&\stackrel{\mathcal{C}}{\sim} \frac{i}{2C}(s_0 - r_0^{(\lambda)})^2 - \frac{1}{2}(s_0 - r_0^{(\lambda)}) - \frac{i}{12}C
\end{aligned}$$

Thus again we have immediately in (35) that

$$r_{\zeta_k}^{(\lambda)}(s_0, -1) = 0 \quad (38)$$

as a function of s_0 , and therefore in (24)

$$r_{\zeta_k}(s_0, -1) = 0 \quad (39)$$

also as a function of s_0 , so that likewise $d_{\zeta_k}(s_0, -1) = r_{\zeta_k}(s_0, -1)$ as expected and the generalised root identity for ζ_k is confirmed when $\mu = -1$.

Case 3 - $\mu = -1$: In this case, after extensive simplification,

$$\begin{aligned}
N_{+,-2}(T) &= \left\{ \begin{array}{l} (s_0 - r_0^{(\lambda)})^2(k+1) - iC(s_0 - r_0^{(\lambda)})(k^2 + k) \\ -C^2(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k) \end{array} \right\} \\
&\stackrel{\mathcal{C}}{\sim} -\frac{i}{3C}(s_0 - r_0^{(\lambda)})^3 + \frac{1}{2}(s_0 - r_0^{(\lambda)})^2 + \frac{i}{12}C(s_0 - r_0^{(\lambda)}) + \frac{1}{12}C\tau_0
\end{aligned}$$

and

$$\begin{aligned}
N_{-,-2}(\tilde{T}) &= \left\{ \begin{array}{l} (s_0 - r_0^{(\lambda)})^2\tilde{k} + iC(s_0 - r_0^{(\lambda)})(\tilde{k}^2 + \tilde{k}) \\ -C^2(\frac{1}{3}\tilde{k}^3 + \frac{1}{2}\tilde{k}^2 + \frac{1}{6}\tilde{k}) \end{array} \right\} \\
&\stackrel{\mathcal{C}}{\sim} \frac{i}{3C}(s_0 - r_0^{(\lambda)})^3 - \frac{1}{2}(s_0 - r_0^{(\lambda)})^2 - \frac{i}{12}C(s_0 - r_0^{(\lambda)}) - \frac{1}{12}C\tau_0
\end{aligned}$$

Hence again in (35) we have immediately that

$$r_{\zeta_k}^{(\lambda)}(s_0, -2) = 0 \quad (40)$$

as a function of s_0 , and therefore in (24)

$$r_{\zeta_k}(s_0, -2) = 0 \quad (41)$$

as a function of s_0 , so that once more $d_{\zeta_k}(s_0, -2) = r_{\zeta_k}(s_0, -2)$ and the generalised root identity for ζ_k is also confirmed when $\mu = -2$.

Overall then we confirm that, for the setting of zeta functions of curves over finite fields, the root identities are satisfied when $\mu = 0, -1$ and -2 . Moreover, it is clear from the general form of such zeta functions in (7) and the results derived above for $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ when $\mu = 0, -1$ and -2 that in this setting the generalised root identities could not be used to detect any possible violation of the RH for such zeta functions, since these results for $r_{\zeta_k}^{(\lambda)}$ apply for arbitrary λ with σ_0 any value in $[0, 1]$.

Of course in fact the RH is known to hold for such zeta functions of curves over finite fields, but the above observation nonetheless demonstrates one way of seeing why this state of affairs is consistent with the generalised root identities, in contrast to the situation for the Riemann zeta function discussed in [1], where the root identity for $\mu = -2$ is argued to imply a contradiction of the RH.

In the next section, we provide a second way of seeing this that more closely identifies this contrast. We re-perform the calculations for $r_{\zeta_k}(s_0, \mu)$ when $\mu = 0, -1$ and -2 in a way that directly mimics the calculation for ζ in [1], by working solely on the line $\Re(s) = \frac{1}{2}$ and considering all the roots simultaneously rather than splitting them into a union of sets of equally spaced roots as just done.

4 A second approach to the Cesaro calculation of $r_{\zeta_k}(s_0, \mu)$ for $\mu = 0, -1$ and -2

For simplicity we shall use the results for $r_{\zeta_k}^{(\lambda)}(s_0, \mu)$ ($\mu = 0, -1, -2$) from the last section just to allow us to ignore the contributions to $r_{\zeta_k}(s_0, \mu)$ from the poles which lie equally spaced on $\Re(s) = 0$ and $\Re(s) = 1$, and thus to concentrate solely on the roots (arising from the factorisation of $P_{2g}(q^{-s})$) to which the RH applies.

We let $N(T)$ be the counting function giving the number of roots of $P_{2g}(q^{-s})$ in the critical strip with imaginary part in $(0, T)$. Since the roots of ζ_k are mirror-symmetric in the real axis, so the number of roots in the critical strip with imaginary part in $(-\tilde{T}, 0)$ is likewise $N(\tilde{T})$. Here, without loss of generality, we are assuming that there are no base roots with imaginary part precisely 0 - if there were they could be ignored, just as we have ignored the poles, as constituting a further discrete set of equi-spaced roots of the sort analyzed in section 3.²

In this case we add $2g$ roots every time T increases by C , so

$$N(T) = \frac{2g}{C}T + S(T) \tag{42}$$

where $S(T)$ is then the periodic, period- C function obtained by periodically extending the function $N(T) - \frac{2g}{C}T$ on $[0, C)$ to all of $[0, \infty)$. In analogy with the notation used in [1] for ζ we denote the divergent piece of $N(T)$ by

²Alternatively we could readily incorporate this case into the analysis in this section, as we could with the poles, but for simplicity we opt not to do this here

$$\tilde{N}(T) = \frac{2g}{C}T \quad (43)$$

and $S(T)$ plays the role analogous to the oscillatory argument of the Riemann zeta function. In the case of ζ_k there is no asymptotically decaying piece analogous to the term $\frac{1}{\pi}\delta(T)$.

Note that $S(0) = S(C) = 0$ and that, since the conjugate of a root of ζ_k is also a root, so $S(t) = -S(C - t)$ on $[0, C]$. Thus S is an odd function about the centre-point of each period and pinned at 0 at the end points.

This information, applied in the breakdown given in (42), is already sufficient to calculate $r_{\zeta_k}(s_0, \mu)$ for $\mu = 0$ and -1 , as follows. As in [1], we treat the functions $N(T)$ and $N(\tilde{T})$ as being functions on the critical line (either as a consequence of the RH for ζ_k or, if this is not to be assumed here, as a reflection of the mirror-symmetry of roots of ζ_k in the critical line); thus we have

$$z = (s_0 - \frac{1}{2}) - iT \quad \text{so} \quad T = i(z - (s_0 - \frac{1}{2})) \quad (44)$$

and

$$\tilde{z} = (s_0 - \frac{1}{2}) + i\tilde{T} \quad \text{so} \quad \tilde{T} = -i(\tilde{z} - (s_0 - \frac{1}{2})) \quad (45)$$

Case 1 - $\mu = 0$: In this case, since all roots have $M_i = 1$, we have

$$\begin{aligned} r_{\zeta_k}(s_0, 0) &= \sum_{\{s_0 - \text{roots } \rho_i \text{ of } \zeta_k\}} (s_0 - \rho_i)^0 = \lim_{z, \tilde{z} \rightarrow \infty} \{N(T) + N(\tilde{T})\} \\ &= \lim_{z, \tilde{z} \rightarrow \infty} \{[\tilde{N}(T) + S(T)] + [\tilde{N}(\tilde{T}) + S(\tilde{T})]\} \end{aligned} \quad (46)$$

Now, since S is an odd function about its mid-point on each period, so $\int_0^C S(t) dt = 0$ and its average value on each period is zero. It follows immediately that

$$\lim_{T \rightarrow \infty} S(T) = \lim_{\tilde{T} \rightarrow \infty} S(\tilde{T}) = 0$$

And since $\lim_{z \rightarrow \infty} z = \lim_{\tilde{z} \rightarrow \infty} \tilde{z} = 0$, so

$$\lim_{z \rightarrow \infty} \tilde{N}(T) = \lim_{z \rightarrow \infty} \frac{2g}{C}i(z - (s_0 - \frac{1}{2})) = -\frac{2g}{C}i(s_0 - \frac{1}{2})$$

and

$$\lim_{\tilde{z} \rightarrow \infty} \tilde{N}(\tilde{T}) = \lim_{\tilde{z} \rightarrow \infty} -\frac{2g}{C}i(\tilde{z} - (s_0 - \frac{1}{2})) = \frac{2g}{C}i(s_0 - \frac{1}{2})$$

Combining these results in (46) it follows at once that

$$r_{\zeta_k}(s_0, 0) = 0$$

as claimed before in Section 3 and in agreement with the root identity for ζ_k at $\mu = 0$ and Result 2 for the value of $d_{\zeta_k}(s_0, 0)$.

Case 2 - $\mu = -1$: In this case, since the roots $\rho_i = \beta_i + i\gamma_i$ of ζ_k are mirror-symmetric in the critical line, we have, on invoking the result just proved for $r_{\zeta_k}(s_0, 0)$, that

$$\begin{aligned}
r_{\zeta_k}(s_0, -1) &= - \sum_{\{s_0\text{-roots } \rho_i \text{ of } \zeta_k\}} (s_0 - \rho_i)^1 \\
&= -(s_0 - \frac{1}{2})r_{\zeta_k}(s_0, 0) + i \cdot \sum_{\{s_0\text{-roots } \rho_i \text{ of } \zeta_k\}} \gamma_i \\
&= i \cdot \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \int_0^T t \, dN(t) - \int_0^{\tilde{T}} \tilde{t} \, dN(\tilde{t}) \right\}
\end{aligned}$$

Using the notation from [1] under which $N_i(T)$ represents the i 'th integral of $N(T)$ and similarly for $\tilde{N}(T)$ and $S(T)$, this becomes

$$\begin{aligned}
&r_{\zeta_k}(s_0, -1) \\
&= i \cdot \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \{TN(T) - N_1(T)\} - \{\tilde{T}\tilde{N}(\tilde{T}) - N_1(\tilde{T})\} \right\} \\
&= i \cdot \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \left\{ \begin{array}{l} \{T\check{N}(T) - \check{N}_1(T)\} \\ + \{TS(T) - S_1(T)\} \end{array} \right\} - \left\{ \begin{array}{l} \{\tilde{T}\check{N}(\tilde{T}) - \check{N}_1(\tilde{T})\} \\ + \{\tilde{T}S(\tilde{T}) - S_1(\tilde{T})\} \end{array} \right\} \right\} \quad (47)
\end{aligned}$$

Now, as before,

$$T\check{N}(T) = \frac{2g}{C}T^2 \xrightarrow{C} -\frac{2g}{C}(s_0 - \frac{1}{2})^2$$

and

$$\check{N}_1(T) = \frac{g}{C}T^2 \xrightarrow{C} -\frac{g}{C}(s_0 - \frac{1}{2})^2$$

so

$$T\check{N}(T) - \check{N}_1(T) \xrightarrow{C} -\frac{g}{C}(s_0 - \frac{1}{2})^2$$

and in identical fashion

$$\tilde{T}\check{N}(\tilde{T}) - \check{N}_1(\tilde{T}) \xrightarrow{C} -\frac{g}{C}(s_0 - \frac{1}{2})^2.$$

As for the terms involving $S(T)$, since $\int_0^C S(t) \, dt = 0$, so $S_1(T)$ is also a periodic function with period C satisfying $S_1(0) = S_1(C) = 0$. As such

it converges in a generalised Cesaro sense, under a single application of the averaging operator P , to its average value on each period. Unlike $S(T)$, however, since it is not odd on each period, this average value, S_1^{av} , is not zero; i.e. overall

$$\lim_{T \rightarrow \infty} S_1(T) = \lim_{\tilde{T} \rightarrow \infty} S_1(\tilde{T}) = S_1^{av}$$

where

$$S_1^{av} = \frac{1}{C} \int_0^C S_1(t) dt \neq 0$$

This leaves only the terms $TS(T)$ and $\tilde{T}S(\tilde{T})$. These both have generalised Cesaro limit zero, i.e.

$$\lim_{T \rightarrow \infty} TS(T) = \lim_{\tilde{T} \rightarrow \infty} \tilde{T}S(\tilde{T}) = 0$$

To see this we again simply apply P and use integration by parts, but taking care in doing so to use the unique anti-derivative of $S(t)$ having average value 0, namely $S_1(t) - S_1^{av}$. Specifically

$$\begin{aligned} P[tS(t)](T) &= \frac{1}{T} \int_0^T tS(t) dt \\ &= (S_1(T) - S_1^{av}) - P[S_1(t) - S_1^{av}](T) \end{aligned}$$

Since clearly $P[S_1(t) - S_1^{av}](T) \rightarrow 0$ as $T \rightarrow \infty$, so $P^2[tS(t)](T) \rightarrow 0$ as $T \rightarrow \infty$ and thus $\lim_{T \rightarrow \infty} TS(T) = 0$ as claimed; and likewise for $\lim_{\tilde{T} \rightarrow \infty} \tilde{T}S(\tilde{T})$.

Combining these calculations in (47), we finally deduce that

$$r_{\zeta_k}(s_0, -1) = 0$$

once again, as claimed before in section 3 and in line with the root identity for ζ_k at $\mu = -1$ and Result 2 for the value of $d_{\zeta_k}(s_0, -1)$.

Case 3 - $\mu = -2$: In this case, writing $\rho_i = \beta_i + i\gamma_i = (\frac{1}{2} + \epsilon_i) + i\gamma_i$ as in [1], and again invoking the mirror symmetry of roots of ζ_k in the critical line and the results just derived for $r_{\zeta_k}(s_0, 0)$ and $r_{\zeta_k}(s_0, -1)$, we obtain

$$\begin{aligned}
r_{\zeta_k}(s_0, -2) &= \sum_{\{s_0 - \text{roots } \rho_i \text{ of } \zeta_k\}} (s_0 - \rho_i)^2 \\
&= \sum_{\{s_0 - \text{roots } \rho_i \text{ of } \zeta_k\}} \left((s_0 - \frac{1}{2}) - \epsilon_i - i\gamma_i \right)^2 \\
&= \left\{ \begin{aligned} &(s_0 - \frac{1}{2})^2 r_{\zeta_k}(s_0, 0) + X_\epsilon - 2(s_0 - \frac{1}{2}) r_{\zeta_k}(s_0, -1) \\ &- \sum_{\{s_0 - \text{roots } \rho_i \text{ of } \zeta_k\}} \gamma_i^2 \end{aligned} \right\} \\
&= X_\epsilon - \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \int_0^T t^2 dN(t) + \int_0^{\tilde{T}} \tilde{t}^2 dN(\tilde{t}) \right\} \quad (48)
\end{aligned}$$

where $X_\epsilon = \sum_{\{s_0 - \text{roots } \rho_i \text{ of } \zeta_k\}} \epsilon_i^2$ as per the notation of [1] (and thus X_ϵ is trivially zero under the RH for ζ_k).

Thus, on integration by parts,

$$\begin{aligned}
&r_{\zeta_k}(s_0, -2) \\
&= X_\epsilon - \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \begin{aligned} &\{T^2 N(T) - 2TN_1(T) + 2N_2(T)\} \\ &+ \{\tilde{T}^2 N(\tilde{T}) - 2\tilde{T}N_1(\tilde{T}) + 2N_2(\tilde{T})\} \end{aligned} \right\} \\
&= X_\epsilon - \underset{z, \tilde{z} \rightarrow \infty}{Clim} \left\{ \begin{aligned} &\{T^2 \check{N}(T) - 2T\check{N}_1(T) + 2\check{N}_2(T)\} \\ &+ \{T^2 S(T) - 2TS_1(T) + 2S_2(T)\} \\ &+ \{\tilde{T}^2 \check{N}(\tilde{T}) - 2\tilde{T}\check{N}_1(\tilde{T}) + 2\check{N}_2(\tilde{T})\} \\ &+ \{\tilde{T}^2 S(\tilde{T}) - 2\tilde{T}S_1(\tilde{T}) + 2S_2(\tilde{T})\} \end{aligned} \right\} \quad (49)
\end{aligned}$$

For this case, however, unlike for $\mu = 0$ and $\mu = -1$, we are not yet in a position to complete the calculation of $r_{\zeta_k}(s_0, -2)$, in particular because for this case we need to calculate S_1^{av} (and then calculate for S_2).

To proceed, therefore, we note that it suffices simply to consider the case in which $g = 1$ and there are precisely two distinct base roots with imaginary parts at κ and $C - \kappa$ in the initial period $0 < T < C$ on the critical line. This suffices because the case of general $N(T)$ arising from $2g$ base roots in $0 \leq T < C$ (either with $2g$ distinct steps of height 1 or possibly fewer steps, some of height greater than 1 arising from base roots with the same imaginary part) can clearly be made up as a linear combination of such simpler cases, and since the expression (48) is clearly linear in $N(T)$ and $N(\tilde{T})$, so the value of

$r_{\zeta_k}(s_0, -2)$ in the general case is just the same linear combination of the values of $r_{\zeta_k}(s_0, -2)$ for these simpler cases.³

We thus now assume without loss of generality that $g = 1$ and $N(T)$ has precisely two distinct steps of height 1, at κ and $C - \kappa$ on the interval $0 \leq T < C$ on the critical line, i.e. $N(t)$ has the form

$$N(t) = \begin{cases} 0 & , 0 \leq t < \kappa \\ 1 & , \kappa \leq t < C - \kappa \\ 2 & , C - \kappa \leq t \leq C \end{cases} \quad (50)$$

on $0 \leq t \leq C$. Thus, writing $T = Ck + \alpha$, $0 \leq \alpha < C$ (and $\tilde{T} = C\tilde{k} + \tilde{\alpha}$) in the usual fashion, we have

$$\tilde{N}(T) = \frac{2}{C}T \quad (51)$$

and $S(T)$ is the periodic, period- C function with the properties described earlier given by

$$S(T) = -\frac{2}{C}\alpha + \begin{cases} 0 & , 0 \leq \alpha < \kappa \\ 1 & , \kappa \leq \alpha < C - \kappa \\ 2 & , C - \kappa \leq \alpha \leq C \end{cases} \quad (52)$$

Hence $S_1(T)$ is given by

$$S_1(T) = -\frac{1}{C}\alpha^2 + \begin{cases} 0 & , 0 \leq \alpha < \kappa \\ \alpha - \kappa & , \kappa \leq \alpha < C - \kappa \\ 2\alpha - C & , C - \kappa \leq \alpha \leq C \end{cases} \quad (53)$$

and an elementary computation then shows that in this case

$$S_1^{av} = \frac{1}{C} \int_0^C S_1(t) dt = \frac{1}{C}\kappa^2 - \kappa + \frac{1}{6}C \quad (54)$$

Taking the terms in (49) in turn then we have, in the usual way, that

$$T^2 \tilde{N}(T) = \frac{2}{C}T^3 \xrightarrow{C} \frac{2}{C}i(s_0 - \frac{1}{2})^3$$

$$T \tilde{N}_1(T) = \frac{1}{C}T^3 \xrightarrow{C} \frac{1}{C}i(s_0 - \frac{1}{2})^3$$

$$\tilde{N}_2(T) = \frac{1}{3C}T^3 \xrightarrow{C} \frac{1}{3C}i(s_0 - \frac{1}{2})^3$$

³Note also that the case of $g = 1$, but with $N(T)$ having a single step of height 2 at $\frac{C}{2}$, follows as a limiting case of the one we consider in which $\kappa \rightarrow \frac{C}{2}$; so in treating only this single simple case with distinct jumps at κ and $C - \kappa$ we are indeed calculating all that is required in order to cover the general case.

and similarly

$$\begin{aligned}\tilde{T}^2 \check{N}(\tilde{T}) &\stackrel{C}{\rightarrow} -\frac{2}{C} i(s_0 - \frac{1}{2})^3 \\ \tilde{T} \check{N}_1(\tilde{T}) &\stackrel{C}{\rightarrow} -\frac{1}{C} i(s_0 - \frac{1}{2})^3 \\ \check{N}_2(\tilde{T}) &\stackrel{C}{\rightarrow} -\frac{1}{3C} i(s_0 - \frac{1}{2})^3\end{aligned}$$

Thus

$$\lim_{z, \tilde{z} \rightarrow \infty} \left\{ \begin{aligned} &\{T^2 \check{N}(T) - 2T \check{N}_1(T) + 2\check{N}_2(T)\} \\ &+ \{\tilde{T}^2 \check{N}(\tilde{T}) - 2\tilde{T} \check{N}_1(\tilde{T}) + 2\check{N}_2(\tilde{T})\} \end{aligned} \right\} = 0 \quad (55)$$

As for the terms in (49) involving $S(T)$, since $S_1^{av} \neq 0$, so $S_2(T)$ is no longer periodic and thus requires the removal of a divergent piece before averaging in order to find its Cesaro limit. Since we still have $\alpha^n \stackrel{C}{\sim} \frac{C^n}{n+1}$ as in section 3, and thus also

$$Ck = T - \alpha \stackrel{C}{\sim} -i(s_0 - \frac{1}{2}) - \frac{1}{2}C$$

and likewise

$$C\tilde{k} = \tilde{T} - \tilde{\alpha} \stackrel{C}{\sim} i(s_0 - \frac{1}{2}) - \frac{1}{2}C$$

so integrating the expression for $S_1(T)$ we have

$$\begin{aligned}S_2(T) &= S_1^{av} \cdot Ck + \int_0^\alpha S_1(t) dt \\ &= S_1^{av} \cdot Ck - \frac{1}{3C} \alpha^3 + Q(\alpha) \\ &\stackrel{C}{\sim} \{-i(s_0 - \frac{1}{2}) - \frac{1}{2}C\} S_1^{av} - \frac{1}{12} C^2 + Q(\alpha)\end{aligned} \quad (56)$$

where Q is the (discontinuous), periodic, period- C function given by

$$Q(T) = Q(\alpha) = \begin{cases} 0 & , 0 \leq \alpha < \kappa \\ \frac{1}{2}(\alpha - \kappa)^2 & , \kappa \leq \alpha < C - \kappa \\ (\alpha - \frac{1}{2}C)^2 + (\kappa - \frac{1}{2}C)^2 & , C - \kappa \leq \alpha \leq C \end{cases}$$

Since Q is periodic, its Cesaro limit is its average value, which in light of (54) is given by

$$\begin{aligned}
Q^{av} &= \frac{1}{C} \int_0^C Q(\alpha) d\alpha \\
&= \frac{1}{C} \left\{ \frac{1}{6}(C-2\kappa)^3 + \frac{1}{3} \left\{ \frac{1}{8}C^3 - \left(\frac{1}{2}C - \kappa\right)^3 \right\} + \left(\kappa - \frac{1}{2}C\right)^2 \kappa \right\} \\
&= \frac{1}{6}C^2 - \frac{1}{2}C\kappa + \frac{1}{2}\kappa^2 = \frac{1}{2}CS_1^{av} + \frac{1}{12}C^2
\end{aligned} \tag{57}$$

Combining (57) in (56) we thus have

$$Clim_{z \rightarrow \infty} S_2(T) = -i(s_0 - \frac{1}{2})S_1^{av} = -i(s_0 - \frac{1}{2}) \left\{ \frac{1}{C}\kappa^2 - \kappa + \frac{1}{6}C \right\}$$

and likewise, by an identical computation,

$$Clim_{\tilde{z} \rightarrow \infty} S_2(\tilde{T}) = i(s_0 - \frac{1}{2})S_1^{av} = i(s_0 - \frac{1}{2}) \left\{ \frac{1}{C}\kappa^2 - \kappa + \frac{1}{6}C \right\}$$

We next calculate the Cesaro limit of $TS_1(T)$ (and its counterpart $\tilde{T}S_1(\tilde{T})$); writing this as

$$TS_1(T) = S_1^{av}T + T(S_1(T) - S_1^{av}) \tag{58}$$

we have, in the usual way,

$$S_1^{av}T \xrightarrow{C} -i(s_0 - \frac{1}{2})S_1^{av} \tag{59}$$

and it remains only to calculate the Cesaro limit of the second term in (58). Since $S_1(T) - S_1^{av}$ is periodic with average value 0 its Cesaro limit is obtained from application of a pure power of P , without need for removal of any further divergences. Defining $\check{S}_2(T) := \int_0^T (S_1(t) - S_1^{av}) dt = S_2(T) - S_1^{av}T$, on integration by parts we have

$$\begin{aligned}
P[t(S_1(t) - S_1^{av})](T) &= \frac{1}{T} \int_0^T t(S_1(t) - S_1^{av}) dt \\
&= \check{S}_2(T) - P[\check{S}_2](T)
\end{aligned} \tag{60}$$

But the fact that $S_1(T) - S_1^{av}$ is periodic with average value 0 means that $\check{S}_2(T)$ must also be periodic, with Cesaro limit given by its average value in each period, \check{S}_2^{av} , i.e.

$$P[\check{S}_2](T) \rightarrow \check{S}_2^{av} \quad \text{as} \quad T \rightarrow \infty$$

It follows immediately in (60) that

$$P^2[t(S_1(t) - S_1^{av})](T) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \quad (61)$$

and so

$$T(S_1(T) - S_1^{av}) \stackrel{\mathcal{C}}{\sim} 0 \quad (62)$$

Combining (59) and (62) in (58) it follows that

$$\lim_{z \rightarrow \infty} T S_1(T) = -i(s_0 - \frac{1}{2})S_1^{av}$$

and likewise, by an identical computation,

$$\lim_{\tilde{z} \rightarrow \infty} \tilde{T} S_1(\tilde{T}) = i(s_0 - \frac{1}{2})S_1^{av}$$

We now lastly calculate the Cesaro limit of $T^2 S(T)$ (and its counterpart $\tilde{T}^2 S(\tilde{T})$). Using $S_1(T) - S_1^{av}$ as an antiderivative of $S(T)$ in integration by parts, we have

$$\begin{aligned} P[t^2 S(t)](T) &= \frac{1}{T} \int_0^T t^2 S(t) dt \\ &= T(S_1(T) - S_1^{av}) - 2P[t(S_1(t) - S_1^{av})](T) \end{aligned}$$

But then it follows directly from (61) that

$$P^3[t^2 S(t)](T) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty$$

and so

$$\lim_{T \rightarrow \infty} T^2 S(T) = 0$$

without any need to remove divergences prior to averaging.⁴By identical reasoning we likewise have

$$\lim_{\tilde{T} \rightarrow \infty} \tilde{T}^2 S(\tilde{T}) = 0$$

Finally, combining all our results for $S(T)$ -related terms in (49), we have

$$\lim_{z, \tilde{z} \rightarrow \infty} \left\{ \begin{aligned} &\{T^2 S(T) - 2T S_1(T) + 2S_2(T)\} \\ &+ \{\tilde{T}^2 S(\tilde{T}) - 2\tilde{T} S_1(\tilde{T}) + 2S_2(\tilde{T})\} \end{aligned} \right\} = 0 \quad (63)$$

⁴Hence the use of the notation $\lim_{T \rightarrow \infty}$ rather than $\lim_{z \rightarrow \infty}$, as we have done at times earlier. The lack of need to remove divergences reflects the fact that $S(T)$ is odd about its mid-point on each period, so that $T^2 S(T)$ is essentially pure oscillatory, albeit of increasing amplitude as $T \rightarrow \infty$

(in fact each of the two pieces in this expression is itself zero) and combining (55) and (63) in (49) it follows at last that

$$r_{\zeta_k}(s_0, -2) = X_\epsilon \quad (64)$$

It follows at once from Result 2 for $d_{\zeta_k}(s_0, -2)$ and the root identity for ζ_k at $\mu = -2$ that we must have

$$X_\epsilon = 0 \quad (65)$$

Thus we see at once that, unlike for ζ in [1], there is no contradiction between the $\mu = -2$ root identity and the RH for ζ_k , since under the RH for ζ_k all $\epsilon_i = 0$ and so X_ϵ is trivially zero. This is in contrast to the case for ζ , where we had $d_\zeta(s_0, -2) = 0$ but $r_\zeta(s_0, -2) = -\frac{1}{2} + X_\epsilon$, so that the root identity for ζ at $\mu = -2$ implies $X_\epsilon = \frac{1}{2} \neq 0$ and thus we cannot have $\epsilon_i = 0$ for all roots of ζ as required by the RH for ζ .

Note, however, that the fact that the $\mu = -2$ root identity implies $X_\epsilon = 0$ in the case of ζ_k is not sufficient alone to actually *deduce* the RH for such zeta functions of curves over finite fields. This is because, as we have shown in section 3, it is possible to have infinitely many roots off the critical line, but still have the Cesaro limit defining X_ϵ for these roots be zero - for example if all the roots of this form lie equi-spaced in a vertical line with $\Re(s) = \sigma_0 \neq \frac{1}{2}$, or comprise a union of such sets of roots.

Thus deduction of the RH for zeta functions of curves over finite fields requires more than just the root identities for $\mu = 0, -1$ and -2 , and indeed it requires work reaching down to algebraic considerations concerning the field k of a sort which have not arisen in the root identity computations here.

5 Summary and Observations

Overall we have shown that for zeta functions of curves over finite fields, ζ_k , the generalised root identities (1) are satisfied for arbitrary s_0 ($\Re(s_0) > 1$) and μ and we have calculated the detailed implications of this for $\mu = 0, -1$ and -2 in two independent ways.

The first approach, in section 3, uses the fact that the generalised roots of such ζ_k form equi-spaced sets on vertical lines $\Re(s) = \sigma_0$, $0 \leq \sigma_0 \leq 1$, to deduce directly that we do have $d_{\zeta_k}(s_0, \mu) = r_{\zeta_k}(s_0, \mu) = 0$ for $\mu = 0, -1$ and -2 .⁵

The second approach in section 4 directly mimics the calculational approach used for the Riemann zeta function in [1] and shows that we have $d_{\zeta_k}(s_0, \mu) = r_{\zeta_k}(s_0, \mu) = 0$ when $\mu = 0$ or -1 , while $d_{\zeta_k}(s_0, -2) = 0$ and $r_{\zeta_k}(s_0, -2) = X_\epsilon$, so that we must have $X_\epsilon = 0$. This is certainly consistent with the RH for ζ_k , which is famously true, in contrast to the situation for ζ in [1] where the $\mu = -2$ root identity is seen to imply a contradiction of the RH. As noted in section 4,

⁵And in fact this will continue to be true for $\mu = -3, -4, \dots$ since these sort of bidirectional equally-spaced remainder sums generally give 0 as their Cesaro sums for elementary, entire sum-functions

however, the fact that $X_\epsilon = 0$ is not sufficient to actually deduce the RH for ζ_k on its own.

We conclude with a brief discussion of why the implications of the generalised root identities for ζ_k and ζ differ (the RH is true for ζ_k but appears false for ζ on the basis of the $\mu = -2$ root identity) even though the zeta functions in both settings satisfy both Euler and Hadamard product formulas.

The reason appears to be that in the case of zeta functions of curves over finite fields the detailed structure of the counting function $N(T)$ counting non-trivial roots in the critical strip is much simpler than it is for ζ . Specifically:

(a) For ζ_k , $N(T)$ has only divergent and oscillatory pieces, $\tilde{N}(T)$ and $S(T)$, but no decaying asymptotic piece analogous to $\frac{1}{\pi}\delta(T)$ in the corresponding breakdown of $N(T)$ as $\tilde{N}(T) + S(T) + \frac{1}{\pi}\delta(T)$ for ζ . The asymptotic piece $\frac{1}{\pi}\delta(T)$ makes a non-trivial contribution to the root side of the $\mu = -2$ root identity in the case of ζ .

(b) The form of the divergent piece $\tilde{N}(T) = \frac{2g}{C}T$ for ζ_k is much simpler than the corresponding divergent piece $\tilde{N}(T) = \frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}$ for ζ , reflecting the much simpler equi-spacing of the generalised roots for each λ -factor in the general factorisation of ζ_k in (7). In particular, the extra $\ln\left(\frac{T}{2\pi}\right)$ factor in $\tilde{N}(T)$ for ζ plays a pivotal role in the root-side calculations for $r_\zeta(s_0, \mu)$ for $\mu = 0, -1$ and -2 , since it leads to non-trivial differences in Cesaro limiting behaviour as $T \rightarrow \infty$ and as $\tilde{T} \rightarrow \infty$, and thereby leads to the non-zero contributions to $r_\zeta(s_0, \mu)$ from the non-trivial roots in these cases. By contrast, for ζ_k , the contributions from the $\tilde{N}(T)$ pieces as $T \rightarrow \infty$ and as $\tilde{T} \rightarrow \infty$ always cancel fully, leading to zero overall contributions from this divergent piece to $r_{\zeta_k}(s_0, \mu)$ when $\mu = 0, -1$ or -2 .

(c) For ζ_k the structure of the oscillatory piece, $S(T)$, is much simpler than it is for the corresponding argument of the Riemann zeta function in the case of ζ . Consequently, in the case of ζ_k , it is easy to calculate and estimate $S(T)$ and its integrals, $S_i(T)$, in contrast to the situation for $S(T)$ and the $S_i(T)$ for ζ , where we had to rely in [1] on established but difficult estimates to obtain Cesaro limits for these functions, and where these estimates were only unconditional in the cases of S and S_1 , being conditional on the RH for ζ for the higher S_i , $i \geq 2$. Note, however, that the contributions from $S(T)$ -related terms to the root sides of the root identities when $\mu = 0, -1$ and -2 is zero in both settings (at least modulo the RH in the case of ζ when $\mu = -1$ or -2).

Overall then, we see from the results of this paper, and from the comparison between the root identity computations in the setting of the Riemann zeta function (in [1]) and of zeta functions of curves over finite fields here, that it is not merely the existence of both Euler and Hadamard product formulas that determines the consequences of these root identities (especially when $\mu = 0, -1$ and -2). The detailed, explicit form of the function $N(T)$ is also crucial in driving the Cesaro computations on the root sides of these identities; and in turn is thus what drives the apparent contradiction of the RH in the case of ζ , while remaining consistent with the RH for ζ_k .

6 Appendices

6.1 Tools for performing numerical calculations used in this paper

The following is the R-code used to generate the example graphs in section 2, which demonstrate numerically the fact that zeta functions of curves over finite fields, ζ_k , do indeed satisfy the generalised root identities (result 1). It is straightforward to adapt this code to conduct further testing as desired.

```
Code: #####
# set working directory (adapt as appropriate the argument of the setwd function
# for setting the working directory in the first line of code)
#####
setwd("C:/...")
options(stringsAsFactors=FALSE)
#####
# Derivative side for given lambda-factor (ignore nu_lambda terms as
# same on both sides)
#####
DerivSide <- function(s0, mu, q, sigma0, tau0, Nterms) {
  theta0 <- tau0 * log(q)
  lambda <- (q^sigma0)* exp(complex(real=0,imaginary=theta0))
  p1 <- exp(complex(real=0,imaginary=pi * mu))/gamma(mu)
  f1 <- lambda * (log(q))^mu * (q^(-s0))
  for (n in 2:Nterms) {
    tp <- ((lambda^n)/(n^(1-mu)))*((log(q))^mu) * (q^(-n*s0))
    f1 <- f1+tp
  }
  p1*(sum(f1))
}
#####
# Root side for given lambda factor (ignore nu_lambda terms as same
# on both sides)
#####
RootSide <- function(s0, mu, q, sigma0, tau0, Nroots) {
  r0_lambda <- complex(real=sigma0,imaginary=tau0)
  C <- 2*pi/log(q)
  rootsVec <- 1:Nroots
  resA1 <- sum(complex(real=rep(s0-sigma0, Nroots), imaginary=-tau0
-C*rootsVec)^(-mu))
  resA2 <- sum(complex(real=rep(s0-sigma0, Nroots), imaginary=-tau0
+ C*rootsVec)^(-mu))
  resA <- resA1+resA2+(complex(real=s0-sigma0, imaginary=-tau0)^(-mu))
  resA<-resA-complex(real=0,imaginary=1)*(1/((1-mu)*C))*complex(real=s0-sigma0,
imaginary=-tau0-C*Nroots)^(1-mu) + complex(real=0,imaginary=1)*(1/((1-mu)*C))
*complex(real=s0-sigma0, imaginary=-tau0+C*Nroots)^(1-mu)
  resA <- resA - 0.5*complex(real=s0-sigma0, imaginary=-tau0-C*Nroots)^(-mu)
- 0.5*complex(real=s0-sigma0, imaginary=-tau0+C*Nroots)^(-mu)
```

```

resA<-resA-complex(real=0,imaginary=1)*mu*(C/12)*complex(real=s0-sigma0,
imaginary=-tau0-C*Nroots)^(-1-mu) + complex(real=0,imaginary=1)*mu*(C/12)
*complex(real=s0-sigma0, imaginary=-tau0+C*Nroots)^(-1-mu)
resA<-resA-complex(real=0,imaginary=1)*mu*(mu+1)*(mu+2)*((C^3)/720)*
complex(real=s0-sigma0,imaginary=-tau0-C*Nroots)^(-3-mu)
+ complex(real=0,imaginary=1)*mu*(mu+1)*(mu+2)*((C^3)/720)
*complex(real=s0-sigma0, imaginary=-tau0+C*Nroots)^(-3-mu)
exp(complex(real=0, imaginary=pi*mu))*resA
}
#####
# Graph mu > -1.5
#####
s0 <- 5.1238
muVec <- seq(-1.45, 2.55, .1)
q <- 25
sigma0 <- 0.6
tau0 <- 3*pi/4
Nterms <- 20
Nroots <- 1000
derivVec <- unlist(lapply(muVec, FUN=function(mu){DerivSide(s0, mu, q, sigma0,
tau0, Nterms)}))
rootVec <- unlist(lapply(muVec, FUN=function(mu){RootSide(s0, mu, q, sigma0,
tau0, Nroots)}))
png("Full Test Re Root and Deriv side mu geq -1.5 1000.png")
plot(muVec, Re(derivVec), col="blue", main="Re Root and Derivative side ZetaFF
(mu>-1.5)", sub="1000 roots")
points(muVec, Re(rootVec), col="red", pch=20)
legend("topright", fill=c("blue","red"), legend=c("Re Deriv side", "Re Root side"))
abline(h=0)
dev.off()
png("Full Test Im Root and Deriv side mu geq -1.5 1000.png")
plot(muVec, Im(derivVec), col="blue", main="Im Root and Derivative side ZetaFF
(mu>-1.5)", sub="1000 roots")
points(muVec, Im(rootVec), col="red", pch=20)
legend("topright", fill=c("blue","red"), legend=c("Im Deriv side", "Im Root side"))
abline(h=0)
dev.off()

```

7 Acknowledgements

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